

# On the monodromy problem for the four-punctured sphere

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## Abstract

We consider the monodromy problem for the four-punctured sphere in which the character of one composite monodromy is fixed, by looking at the expansion of the accessory parameter in the modulus  $x$  directly, without taking the limit of the quantum conformal blocks for infinite central charge. The integrals which appear in the expansion of the Volterra equation, involve products of two hypergeometric functions to first order and up to four hypergeometric functions to second order. It is shown that all such integrals can be computed analytically. We give the complete analytical evaluation of the accessory parameter to first and second order in the modulus. The results agree with the evaluation obtained by assuming the exponentiation hypothesis of the quantum conformal blocks in the limit of infinite central charge. Extension to higher orders is discussed.

# 1 Introduction

In the papers [1, 2] the following monodromy problem is considered for the Liouville theory on the sphere: Given the singularities in the standard position  $0, x, 1, \infty$  and given the class i.e. the trace of the monodromy for a path encircling both singularities at 0 and  $x$ , find the value of the accessory parameter realizing such data. Such a problem intervenes in the process of the classical limit of the quantum four-point function; the value of trace of the above described monodromy is then fixed by a saddle point procedure [3]. In [3, 1] the problem is solved by going over to the quantum formulation for the four-point function and taking the classical limit i.e. the limit in which the central charge goes to infinity. As the quantum conformal blocks are known as formal power series expansions in  $x$  also the classical result so obtained is given as a formal power expansion in  $x$ . The procedure goes through a process of exponentiation of the quantum conformal blocks after which the classical limit  $b \rightarrow 0$  is taken; in such a limit heavy cancellations take part [4]. Several analytic [5, 6, 1, 7] and numerical [3, 4] calculations, also exploiting recursion formulae [8, 9] for the conformal blocks, support the validity of such a calculational scheme. One suspects, on the other hand, that the same results should be obtainable just by exploiting the transformation properties of the ordinary differential equations which underlie the Liouville theory at the classical level. In this note we shall in fact consider the problem directly at the classical level. In addition it appears that working without taking the singular limit in which the central charge goes to infinity one might control better the convergence region of the expansion of the accessory parameter as a function of the modulus [10].

The approach followed in this paper consists in computing the monodromy along a contour embracing 0 and  $x$  through the usual convergent iteration expansion for the solution of the Heun equation. The monodromy is computed along a contour which avoids the neighborhood of the origin where the kernel is singular and then we expand the result in  $x$ .

In so doing one is faced to first order with the computation of integrals containing the product of two hypergeometric functions; if one goes to the second order the product of four hypergeometric functions in a double integral appears.

In this paper we show how to compute analytically such integrals, which appear in the expansion of the solution of the Volterra equation. The complete first order result gives as a byproduct the value of the accessory parameter which coincides with the one derived in [3] and re-derived in [6, 1].

For computing the integrals appearing in the first order result, we exploit the transforma-

tion property of the solution of the differential equation under  $SL(2, C)$ . In the calculation of the second order such a technique is not sufficient and we need a non invertible transformation which at the infinitesimal level is related to the operator  $l_{-2} = \frac{1}{z} \frac{\partial}{\partial z}$ . Contrary to the  $SL(2, C)$  transformations this is not one-to-one in the complex plane.

On the other hand the procedure we shall describe, involves only the solutions along the real  $z$  axis for  $z \geq 1$ , and there for  $|x| < 1$  the transformation is well defined.

After developing such tools we give the complete second order computation for the accessory parameter. In so doing we employ a formalism apt to be extended to higher order computation.

The second order result agrees with the one obtained in [6, 1] by considering the classical limit of the quantum conformal blocks combined with the exponentiation hypothesis and thus it lends a strong support to the exponentiation hypothesis of the conformal blocks in the  $b \rightarrow 0$  limit. We discuss also the extension of the procedure to higher orders.

Obviously, as the determination of the accessory parameter  $C(x)$  is always obtained through the solution of an implicit equation, the fact that the function  $Q(z)$  which represents the energy momentum tensor, has radius of convergence 1 in  $x$ , for  $z > 1$ , does not assure that the expansion of  $C(x)$  in  $x$  has the same radius of convergence. For achieving rigorous lower bounds on such a radius of convergence, methods similar to those developed in [10, 11] for the convergence in the coupling strength should be applied. The developed technique can also be applied to the problem of the punctured torus.

## 2 General setting

The ordinary differential equation associated with the monodromy problem is

$$y''(z) + Q(z)y(z) = 0 \quad (1)$$

with

$$Q(z) = \frac{\delta_0}{z^2} + \frac{\delta}{(z-x)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{z(z-1)} + \frac{C(x)}{z(z-x)(1-z)} \quad (2)$$

where  $\delta_j = (1 - \lambda_j^2)/4$ .  $C(x)$  is the accessory parameter to be fixed so that the monodromy along a contour encircling both 0 and  $x$  has trace  $-2 \cos \pi \lambda_\nu$  and as such it will depend both on  $x$  and  $\delta_\nu$ . We have

$$C(0) = \delta_\nu - \delta_0 - \delta \quad (3)$$

and  $C(x)$  is related to to one used in [3, 1] which we call  $C_L(x)$ , by  $C(x) = x(1-x)C_L(x)$  and thus  $C(0) = xC_L(x)|_{x=0}$  and  $C'(0) = [xC_L(x)]'|_{x=0} - C(0)$ .

Expanding in  $x$  we have

$$Q = Q_0 + xQ_1 + x^2Q_2 + O(x^3) \quad (4)$$

$$\begin{aligned} Q_0 &= \frac{\delta_\nu}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_\nu - \delta_1}{z(z-1)} \\ Q_1 &= \frac{2\delta - C'(0)}{z^2(z-1)} - \frac{2\delta + C(0)}{z^3(z-1)} \\ Q_2 &= -\frac{C''(0)}{2z^2(z-1)} + \frac{3\delta - C'(0)}{z^3(z-1)} - \frac{3\delta + C(0)}{z^4(z-1)}. \end{aligned} \quad (5)$$

It is our interest to compute the class of the monodromy along a circuit enclosing both the origin and  $x$ . Working near the origin is difficult due to the singular nature of the kernel. Instead we shall compute the same monodromy along the circuit shown in fig.1. The great advantage in performing such a change in the contour is the fact that the expansion in  $x$  of  $Q(z)$  along the contour is no longer singular and actually is convergent with convergence radius 1.  $Y$  will denote the complex

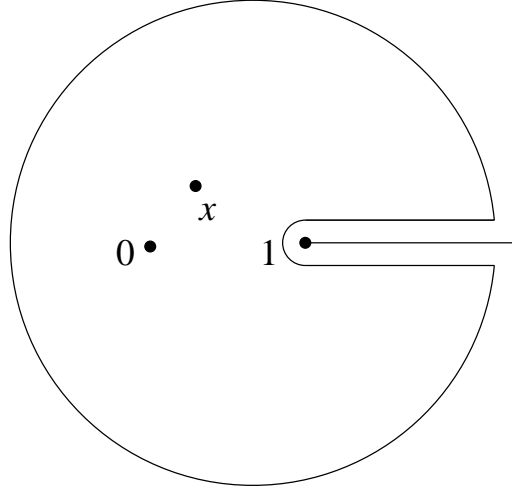


Figure 1: the integration contour

$$Y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix} \quad (6)$$

being  $y_k$  two independent solutions of  $y_k'' + Q_0 y_k = 0$ , canonical at  $z = 1$ . They are given by

$$y_1(z) = (1-z)^{\frac{1-\lambda_1}{2}} z^{\frac{1-\lambda_\nu}{2}} F\left(\frac{1-\lambda_1-\lambda_\infty-\lambda_\nu}{2}, \frac{1-\lambda_1+\lambda_\infty-\lambda_\nu}{2}, 1-\lambda_1; 1-z\right) \quad (7)$$

$$y_2(z) = (1-z)^{\frac{1+\lambda_1}{2}} z^{\frac{1+\lambda_\nu}{2}} F\left(\frac{1+\lambda_1+\lambda_\infty+\lambda_\nu}{2}, \frac{1+\lambda_1-\lambda_\infty+\lambda_\nu}{2}, 1+\lambda_1; 1-z\right). \quad (8)$$

The constant Wronskian is easily computed at  $z = 1$

$$w_{12} = y_1 y_2' - y_1' y_2 = -\lambda_1. \quad (9)$$

The unperturbed monodromy is computed as follows. We start from the  $Y$  for real  $z$ ,  $z < 1$ . The continuation to the upper side of the cut  $(1, +\infty)$  is given by

$$\begin{aligned} y_1^+(z) &= -ie^{\frac{i\pi\lambda_1}{2}} (z-1)^{\frac{1-\lambda_1}{2}} z^{\frac{1-\lambda_\nu}{2}} F\left(\frac{1-\lambda_1-\lambda_\infty-\lambda_\nu}{2}, \frac{1-\lambda_1+\lambda_\infty-\lambda_\nu}{2}, 1-\lambda_1; 1-z\right) \\ &\equiv -ie^{\frac{i\pi\lambda_1}{2}} t_1(z) \end{aligned} \quad (10)$$

$$\begin{aligned} y_2^+(z) &= -ie^{-\frac{i\pi\lambda_1}{2}} (z-1)^{\frac{1+\lambda_1}{2}} z^{\frac{1+\lambda_\nu}{2}} F\left(\frac{1+\lambda_1+\lambda_\infty+\lambda_\nu}{2}, \frac{1+\lambda_1-\lambda_\infty+\lambda_\nu}{2}, 1+\lambda_1; 1-z\right) \\ &\equiv -ie^{-\frac{i\pi\lambda_1}{2}} t_2(z) \end{aligned} \quad (11)$$

whose asymptotic behavior for large  $z$  is

$$Y^+(z) \approx B^+ \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix} = -i \begin{pmatrix} B_2^{(1)} e^{\frac{i\pi\lambda_1}{2}} & B_1^{(1)} e^{\frac{i\pi\lambda_1}{2}} \\ B_1^{(2)} e^{-\frac{i\pi\lambda_1}{2}} & B_2^{(2)} e^{-\frac{i\pi\lambda_1}{2}} \end{pmatrix} \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix} \quad (12)$$

$B_k^{(j)}$  being a well known matrix

$$B_k^{(j)} = \begin{pmatrix} \frac{\Gamma(1-\lambda_1)\Gamma(\lambda_\infty)}{\Gamma(\frac{1-\lambda_1+\lambda_\infty-\lambda_\nu}{2})\Gamma(\frac{1-\lambda_1+\lambda_\infty+\lambda_\nu}{2})} & \frac{\Gamma(1-\lambda_1)\Gamma(-\lambda_\infty)}{\Gamma(\frac{1-\lambda_1-\lambda_\infty-\lambda_\nu}{2})\Gamma(\frac{1-\lambda_1-\lambda_\infty+\lambda_\nu}{2})} \\ \frac{\Gamma(1+\lambda_1)\Gamma(-\lambda_\infty)}{\Gamma(\frac{1+\lambda_1-\lambda_\infty+\lambda_\nu}{2})\Gamma(\frac{1+\lambda_1-\lambda_\infty-\lambda_\nu}{2})} & \frac{\Gamma(1+\lambda_1)\Gamma(\lambda_\infty)}{\Gamma(\frac{1+\lambda_1+\lambda_\infty+\lambda_\nu}{2})\Gamma(\frac{1+\lambda_1+\lambda_\infty-\lambda_\nu}{2})} \end{pmatrix}. \quad (13)$$

Similarly

$$y_1^-(z) = ie^{-\frac{i\pi\lambda_1}{2}} t_1(z), \quad y_2^-(z) = ie^{\frac{i\pi\lambda_1}{2}} t_2(z) \quad (14)$$

and

$$Y^-(z) \approx B^- \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix} = i \begin{pmatrix} B_2^{(1)} e^{-\frac{i\pi\lambda_1}{2}} & B_1^{(1)} e^{-\frac{i\pi\lambda_1}{2}} \\ B_1^{(2)} e^{\frac{i\pi\lambda_1}{2}} & B_2^{(2)} e^{\frac{i\pi\lambda_1}{2}} \end{pmatrix} \begin{pmatrix} z^{\frac{1-\lambda_\infty}{2}} \\ z^{\frac{1+\lambda_\infty}{2}} \end{pmatrix}. \quad (15)$$

We start from  $z = +\infty - i\varepsilon$  eq.(15), whose continuation to the upper side of the cut is eq.(12). Taking the turn of  $2\pi$  at infinity we go back to  $z = +\infty - i\varepsilon$  having encircled the origin and  $x$  and we obtain the monodromy matrix to lowest order

$$M^0 = -(B^+) \begin{pmatrix} e^{-i\pi\lambda_\infty} & 0 \\ 0 & e^{i\pi\lambda_\infty} \end{pmatrix} (B^-)^{-1}. \quad (16)$$

One easily checks that  $\text{tr} M^0 = -2 \cos \pi \lambda_\nu$ . The first order corrections to  $Y$  is provided by

$$Y(z) + x S_1(z) Y(z) \quad (17)$$

with

$$S_1(z) = \frac{1}{w_{12}} \begin{pmatrix} \int_1^z y_2 Q_1 y_1 dz & - \int_1^z y_1 Q_1 y_1 dz \\ \int_1^z y_2 Q_1 y_2 dz & - \int_1^z y_1 Q_1 y_2 dz \end{pmatrix} \quad (18)$$

$w_{12} = -\lambda_1$ . Following the procedure illustrated above using (17) instead of the unperturbed  $Y$  we obtain the new monodromy matrix  $M^0 + \delta M$ ,

$$\delta M = x (S_1^+ M^0 - M^0 S_1^-) \quad (19)$$

with  $S_1 \equiv S_1(\infty)$ , and we have

$$\text{tr} M = -2 \cos \pi \lambda_\nu + x \text{tr}(S_1^+ - S_1^-) M^0 \quad (20)$$

where

$$(S_1^+ - S_1^-)_{12} = \frac{2i \sin \pi \lambda_1}{w_{12}} \int_1^\infty t_1(z) Q_1(z) t_1(z) dz \equiv \frac{2i \sin \pi \lambda_1}{w_{12}} Q_1(1, 1) \quad (21)$$

$$(S_1^+ - S_1^-)_{21} = \frac{2i \sin \pi \lambda_1}{w_{12}} \int_1^\infty t_2(z) Q_1(z) t_2(z) dz \equiv \frac{2i \sin \pi \lambda_1}{w_{12}} Q_1(2, 2) . \quad (22)$$

We have due to eqs.(10,11,14)

$$(S_1^+ - S_1^-)_{11} = -(S_1^+ - S_1^-)_{22} = 0 \quad (23)$$

and thus

$$\text{tr} \delta M = x[(S_1^+ - S_1^-)_{12} M_{21}^0 + (S_1^+ - S_1^-)_{21} M_{12}^0] \quad (24)$$

and to determine  $C'(0)$  we must impose  $\text{tr} \delta M = 0$ .

The second order result is obtained by iterating once the result with  $Q_1$  and adding also the contribution obtained by replacing in (17)  $x Q_1$  with  $x^2 Q_2$ .

### 3 First order calculation

The computation of the first order result is reasonably simple. We shall adopt here a formalism which is apt to be extended to the second and higher order computation.

We are faced to compute the integrals appearing in eqs.(21,22) with  $Q_1(z)$  given in eq.(5).

More generally we shall compute analytically the indefinite integrals

$$\int_1^z \frac{t_j(z) t_k(z)}{z^m (z-1)} dz \quad (25)$$

for  $m \geq 2$  and where  $j$  and  $k$  take the value 1 and 2.

For  $m = 2, 3$  such integrals can be computed by exploiting the transformation properties under  $SL(2, C)$  of the solutions. Let us consider the equation

$$\tilde{t}'' + R \tilde{t} = 0 \quad (26)$$

$$R = \frac{\delta_\nu}{(z-a)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_1 - \delta_\nu}{(z-a)(z-1)}. \quad (27)$$

The solutions of eq.(26) are

$$\tilde{t}_1(z, a) = (z-1)^{\frac{1-\lambda_1}{2}} \left( \frac{z-a}{1-a} \right)^{\frac{1-\lambda_\nu}{2}} F\left( \frac{1-\lambda_1-\lambda_\infty-\lambda_\nu}{2}, \frac{1-\lambda_1+\lambda_\infty-\lambda_\nu}{2}, 1-\lambda_1; \frac{1-z}{1-a} \right) \quad (28)$$

and similarly for  $\tilde{t}_2$ . Then writing

$$\begin{aligned} R &= R_0 + aR_1 + O(a^2) \\ R_0 &= \frac{\delta_\nu}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_1 - \delta_\nu}{z(z-1)} = Q_0, \quad R_1 = \frac{\delta_\infty + \delta_\nu - \delta_1}{z^2(z-1)} - \frac{2\delta_\nu}{z^3(z-1)} \end{aligned} \quad (29)$$

$$\dot{t}_k = \left. \frac{\partial \tilde{t}_k}{\partial a} \right|_{a=0} \quad (30)$$

using

$$\dot{t}_k'' + R_0 \dot{t}_k + R_1 t_k = 0 \quad (31)$$

and  $\tilde{t}(z, 0) = t_k(z)$  we have

$$\int_1^z t_k R_1 t_j dz = t'_k \dot{t}_j - t_k \dot{t}'_j \Big|_1^z = t'_j \dot{t}_k - t_j \dot{t}'_k \Big|_1^z. \quad (32)$$

We find

$$\begin{aligned} R_1(1, 1) &\equiv \int_1^\infty t_1 R_1 t_1 dz = -\lambda_\infty^2 B_1^{(1)} B_2^{(1)} \\ R_1(2, 2) &\equiv \int_1^\infty t_2 R_1 t_2 dz = -\lambda_\infty^2 B_1^{(2)} B_2^{(2)}. \end{aligned} \quad (33)$$

On the other hand the change  $\delta_\nu$  into  $\delta_\nu - \varepsilon$  induces in  $Q_0$  the change

$$Q_0 \rightarrow Q_0 + \delta Q_0, \quad \delta Q_0 = \frac{\varepsilon}{z^2(z-1)} \quad (34)$$

leaving the singularities at  $z = 1$  and at  $z = \infty$  unchanged. This time the related change  $\delta t_k$  is simply given by

$$\delta t_k = \frac{2\varepsilon}{\lambda_\nu} \frac{\partial t_k}{\partial \lambda_\nu} \quad (35)$$

and we can again apply eq.(32) replacing  $\dot{t}_k$  with  $\frac{2}{\lambda_\nu} \frac{\partial t_k}{\partial \lambda_\nu}$ . Combining with (33) it provides us with the integrals of type (37) with  $m = 2$  and with  $m = 3$  appearing in the first order computation, in terms of hypergeometric functions and derivatives thereof.

The integrals appearing in (21,22) have the upper limit infinity, for which the derived formulae (32) also hold. As the asymptotic behavior of the hypergeometric functions are given [12] by simple powers of  $z$  multiplied by gamma functions, we have that such integrals are expressed in terms of the functions  $\Gamma$  and  $\psi$ , where  $\psi(x) = \Gamma'(x)/\Gamma(x)$ . We shall denote by  $N_m$  the expression

$$N_m = \frac{1}{z^m(z-1)}. \quad (36)$$

They form a basis for the the derivative of  $Q(z)$  with respect to  $x$  to any order and we shall set

$$\int_1^\infty t_k \frac{1}{z^m(z-1)} t_j dz = N_m(k, j). \quad (37)$$

Explicitly we find

$$N_2(1, 1) = \frac{\lambda_\infty}{\lambda_\nu} B_1^{(1)} B_2^{(1)} \Psi(\lambda_1, \lambda_\nu, \lambda_\infty) \equiv \frac{\lambda_\infty}{\lambda_\nu} B_1^{(1)} B_2^{(1)} \Psi_1 \quad (38)$$

and

$$N_2(2, 2) = \frac{\lambda_\infty}{\lambda_\nu} B_1^{(2)} B_2^{(2)} \Psi(-\lambda_1, -\lambda_\nu, -\lambda_\infty) \equiv \frac{\lambda_\infty}{\lambda_\nu} B_1^{(2)} B_2^{(2)} \Psi_2 \quad (39)$$

where we defined

$$\begin{aligned} \Psi(\lambda_1, \lambda_\nu, \lambda_\infty) \equiv & \psi\left(\frac{1 - \lambda_1 - \lambda_\infty - \lambda_\nu}{2}\right) - \psi\left(\frac{1 - \lambda_1 + \lambda_\infty - \lambda_\nu}{2}\right) - \\ & \psi\left(\frac{1 - \lambda_1 - \lambda_\infty + \lambda_\nu}{2}\right) + \psi\left(\frac{1 - \lambda_1 + \lambda_\infty + \lambda_\nu}{2}\right). \end{aligned}$$

Moreover in eq.(24) we have

$$M_{12}^0 \det(B^-) = -2 i \sin \pi \lambda_\infty B_1^{(1)} B_2^{(1)} \quad (40)$$

$$M_{21}^0 \det(B^-) = 2 i \sin \pi \lambda_\infty B_1^{(2)} B_2^{(2)}. \quad (41)$$

with  $\det B^- = \lambda_1/\lambda_\infty$ . For future use we report below also the values of  $M_{11}^0$  and  $M_{22}^0$ .

$$M_{11}^0 \det(B^-) = e^{i\pi\lambda_1} (B_1^{(1)} B_1^{(2)} e^{i\pi\lambda_\infty} - B_2^{(1)} B_2^{(2)} e^{-i\pi\lambda_\infty}) \quad (42)$$

$$M_{22}^0 \det(B^-) = e^{-i\pi\lambda_1} (B_1^{(1)} B_1^{(2)} e^{-i\pi\lambda_\infty} - B_2^{(1)} B_2^{(2)} e^{i\pi\lambda_\infty}). \quad (43)$$

The vectors

$$T(1, 1) = (R_1(1, 1), N_2(1, 1)) = B_1^{(1)} B_2^{(1)} \left( -\lambda_\infty^2, \frac{\lambda_\infty}{\lambda_\nu} \Psi_1 \right) \equiv B_1^{(1)} B_2^{(1)} \hat{T}(1, 1) \quad (44)$$

$$T(2, 2) = (R_1(2, 2), N_2(2, 2)) = B_1^{(2)} B_2^{(2)} \left( -\lambda_\infty^2, \frac{\lambda_\infty}{\lambda_\nu} \Psi_2 \right) \equiv B_1^{(2)} B_2^{(2)} \hat{T}(2, 2) \quad (45)$$



are given by

$$T(k, k) = A \begin{pmatrix} N_2(k, k) \\ N_3(k, k) \end{pmatrix} \quad (46)$$

with

$$A = \begin{pmatrix} \delta_\nu + \delta_\infty - \delta_1 & -2\delta_\nu \\ 1 & 0 \end{pmatrix}. \quad (47)$$

$Q_1$ , see eq.(5), in the basis  $N_2, N_3$  is represented by the vector

$$q_1 = (2\delta - C'(0), -2\delta - C(0)) \quad (48)$$

and thus, see eq.(24), the equation for  $C'(0)$  becomes

$$0 = q_1 \cdot A^{-1} \cdot (\hat{T}(1, 1) - \hat{T}(2, 2)) \quad (49)$$

i.e.

$$0 = 2\delta_\nu(2\delta - C'(0)) - (2\delta + C(0))(\delta_\nu + \delta_\infty - \delta_1) = \quad (50)$$

$$-2\delta_\nu(C'(0) + C(0)) - (-\delta_\nu + \delta_\infty - \delta_1)(\delta_\nu - \delta_0 + \delta) \quad (51)$$

giving

$$C'(0) = \frac{(\delta_\nu - \delta_0 + \delta)(\delta_\nu - \delta_\infty + \delta_1)}{2\delta_\nu} - C(0) = [xC_L(x)]'|_{x=0} - C(0) \quad (52)$$

which is the result of [3, 6, 1] obtained by taking the  $b \rightarrow 0$  limit of the conformal blocks.

## 4 Second order calculation

The equation which gives  $C''(0)$  is provided by the vanishing of the coefficient of  $x^2$  in the expansion of

$$\text{tr}(1 + xS_1^+ + x^2S_2^+)M^0(1 + xS_1^- + x^2S_2^-)^{-1} \quad (53)$$

i.e.

$$0 = \text{tr}(S_2^+ - S_2^-)M^0 - \text{tr}(S_1^+ - S_1^-)M^0S_1^- . \quad (54)$$

The second order change in the functions  $y_k$  is given by the direct contribution due to  $Q_2$  and by the second iteration of the contribution of  $Q_1$ .

With regard to the direct contribution we have to compute

$$\int_1^\infty t_k(z)Q_2(z)t_j(z) dz, \quad (55)$$

with  $Q_2(z)$  given in eq.(5), where the new integrals  $N_4(k, j)$  appear. In the basis  $N_2, N_3, N_4$   $Q_2$  is represented by the vector

$$q_2 = (-C'''(0)/2, 3\delta - C''(0), -3\delta - C(0)) . \quad (56)$$

The  $N_4(j, k)$  cannot be computed by performing an  $SL(2, C)$  transformation. We shall exploit the new transformation

$$z = \frac{v - \frac{a}{v}}{1 - a} \quad (57)$$

and use the schwarzian transformation of  $R_0$  and the  $-\frac{1}{2}$ -form nature [13] of the solutions  $t_k$ . The above transformation gives rise to

$$R(v, a) = Q_0\left(\frac{v^2 - a}{v(1 - a)}\right)\left(\frac{dz}{dv}\right)^2 - \{z, v\} \quad (58)$$

where

$$\{z, v\} = -\frac{3a}{(a + v^2)^2} \quad (59)$$

is the Schwarz derivative of the transformation and the new solutions are

$$t_k(v, a) = \frac{1}{\sqrt{1 - a}} \left(\frac{dz}{dv}\right)^{-\frac{1}{2}} t_k\left(\frac{v^2 - a}{v(1 - a)}\right) = \frac{v}{\sqrt{v^2 + a}} t_k\left(\frac{v^2 - a}{v(1 - a)}\right). \quad (60)$$

Reverting to the  $z$ -notation for the variable and denoting with the dot the derivative w.r.t.  $a$  we have

$$R(z, a) = Q_0\left(\frac{z^2 - a}{z(1 - a)}\right)\left(\frac{z^2 + a}{z^2(1 - a)}\right)^2 + \frac{3a}{(a + z^2)^2} \quad (61)$$

with

$$\dot{R}(z, 0) = \frac{\delta_\nu - \delta_1 - \delta_\infty}{z^2(z - 1)} + \frac{3 - 3\delta_1 + 3\delta_\infty + \delta_\nu}{z^3(z - 1)} - \frac{3 + 4\delta_\nu}{z^4(z - 1)}. \quad (62)$$

From eq.(60) we have

$$\dot{t}_k(z, 0) = -\frac{1}{2z^2} t_k(z) + \left(z - \frac{1}{z}\right) t'_k(z). \quad (63)$$

Contrary to the  $SL(2, C)$  transformations the transformation (57) is not one-to-one in the complex plane. Nevertheless for  $|a| < 1$  the transformation is well defined, i.e. non singular along the line  $1 < z < \infty$  which is our range of application.

We can then exploit again the integration formula (32), with  $\dot{t}_k$  replaced by eq.(63) and  $R_1$  by  $\dot{R}$  of eq.(62). One can also verify the correctness of the result by taking explicitly the derivative w.r.t.  $z$  of the obtained formula.

We introduce the three dimensional vectors  $T^r(k, j)$ , with  $r = 1, 2, 3$  which represent the matrix elements of the variation of  $Q_0$  under respectively the transformation of eq.(27)

( $r = 1$ ), the of eq.(34) ( $r = 2$ ) and of the transformation of the above eq.(57) ( $r = 3$ ). We have

$$T(j, k) = A \begin{pmatrix} N_2(j, k) \\ N_3(j, k) \\ N_4(j, k) \end{pmatrix} \quad (64)$$

with

$$A = \begin{pmatrix} \delta_\infty + \delta_\nu - \delta_1 & -2\delta_\nu & 0 \\ 1 & 0 & 0 \\ \delta_\nu - \delta_1 - \delta_\infty & 3 - 3\delta_1 + 3\delta_\infty + \delta_\nu & -3 - 4\delta_\nu \end{pmatrix} \quad (65)$$

and

$$\begin{aligned} T(1, 1) &= B_1^{(1)} B_2^{(1)} (-\lambda_\infty^2, \frac{\lambda_\infty}{\lambda_\nu} \Psi_1, -\lambda_\infty^2) \equiv B_1^{(1)} B_2^{(1)} \hat{T}(1, 1) \\ T(2, 2) &= B_1^{(2)} B_2^{(2)} (-\lambda_\infty^2, \frac{\lambda_\infty}{\lambda_\nu} \Psi_2, -\lambda_\infty^2) \equiv B_1^{(2)} B_2^{(2)} \hat{T}(2, 2) . \end{aligned} \quad (66)$$

The inversion of eq.(64) provides the values of the fundamental matrix elements  $N_m(j, k)$ ,  $m = 2, 3, 4$ . We notice that the procedure can be extended to all values of  $m$  by considering the variation of the equation  $y'' + R_0 y = 0$  under the transformation

$$z = (v - \frac{a}{v^{m-3}})/(1 - a) . \quad (67)$$

In addition due to the structure of the matrices  $A$  such a procedure is purely iterative, i.e. known  $N_m(j, k)$  for  $m = 2, 3, \dots, n$  the computation of  $T^n(j, k)$ , provides directly  $N_{n+1}(j, k)$ .

In the present case we have

$$Q_2(j, k) = q_2 \cdot A^{-1} \cdot T(j, k) . \quad (68)$$

We come now to the second iteration of  $Q_1$ . Given the Green function

$$G(z, z') = \frac{1}{w_{12}} (y_1(z) y_2(z') - y_2(z) y_1(z')) \quad (69)$$

the expression we are confronted with, for the second order change of  $y_k(z)$  is

$$y_1(z) \frac{1}{w_{12}} \int_1^z Q_1(z') y_2(z') \delta^{(1)} y_k(z') dz' - y_2(z) \frac{1}{w_{12}} \int_1^z Q_1(z') y_1(z') \delta^{(1)} y_k(z') dz' \quad (70)$$

with  $\delta^{(1)} y_k$  given by eq.(17). The indefinite integrals appearing in (70) can be obtained from the equation

$$\tilde{y}''(z, a) + \tilde{Q}(z, a) \tilde{y}(z, a) = 0 \quad (71)$$

with

$$\tilde{Q}(z, a) = \frac{\delta_\nu + sa}{(z - ca)^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_\infty - \delta_1 - \delta_\nu - sa}{(z - ca)(z - 1)} \quad (72)$$

where we shall impose

$$\left. \frac{\partial \tilde{Q}(z, a)}{\partial a} \right|_{a=0} = \frac{2\delta - C'(0)}{z^2(z - 1)} - \frac{2\delta + C(0)}{z^3(z - 1)} = Q_1(z) \quad (73)$$

with  $C(0) = \delta_\nu - \delta_0 - \delta$  and  $C'(0)$  given by eq.(52). In order to fit the coefficients of  $N_2, N_3$  in  $Q_1$  we have to allow in principle for two parameters  $c$  and  $s$ . We find

$$c = \frac{\delta + \delta_\nu - \delta_0}{2\delta_\nu}, \quad s = 0. \quad (74)$$

Notice that the transformation leading from  $Q_0(z)$  to  $\tilde{Q}(z, a)$  is of the same type as the one appearing in (26,27) of which we know the solutions. It follows that after replacing  $C(0)$  and  $C'(0)$  in (73) with their values, the simplest method to compute the matrix elements  $Q_1(j, k)$  is to use the transformation (28) with  $a$  replaced by  $ca$ . One can easily prove that

$$y_1(z) + x \left. \frac{\partial \tilde{y}_1(z, a)}{\partial a} \right|_{a=0} \equiv y_1(z) + x \dot{\tilde{y}}_1(z) \quad (75)$$

has the correct boundary condition  $(1 - z)^{(1-\lambda_1)/2}$  with coefficient 1 at  $z = 1$ , as imposed by the solution of the Volterra equation and thus  $x \dot{\tilde{y}}_1(z)$  equals  $\delta^{(1)} y_1$  i.e. it is the first order correction to  $y_1$ . It is expressed in terms of derivatives of the hypergeometric function. The same holds for  $y_2$ . Then the integrals

$$\int_1^z y_k(z') Q_1(z') \delta^{(1)} y_j(z') dz' \quad (76)$$

appearing in (70) can be computed as follows. Using

$$\tilde{Q}(z, a) = Q_0(z) + a Q_1(z) + a^2 \tilde{Q}_2(z) + \dots \quad (77)$$

and

$$\ddot{\tilde{y}}_k'' + 2\tilde{Q}_2(z) y_k(z) + 2Q_1(z) \dot{\tilde{y}}_k(z) + Q_0(z) \ddot{\tilde{y}}_k(z) = 0 \quad (78)$$

we have

$$\int_1^z y_k(z') Q_1(z') \dot{\tilde{y}}_j(z') dz' = -\frac{1}{2} (y_k(z) \ddot{\tilde{y}}_j'(z) - y_k'(z) \ddot{\tilde{y}}_j(z)) \Big|_1^z - \int_1^z y_k(z') \tilde{Q}_2(z') y_j(z') dz' . \quad (79)$$

Notice that  $\tilde{Q}_2$  is not equal to the  $Q_2$  of eq.(5) but it will be expedient for computing the l.h.s. of (79).  $\tilde{Q}_2$  in the base  $N_2, N_3, N_4$ , is represented by the vector

$$\tilde{q}_2 = c^2 (0, \delta_\infty - \delta_1 + 2\delta_\nu, -3\delta_\nu) . \quad (80)$$

We know  $\ddot{y}_k(z)$  and  $\ddot{y}'_l(z)$  and as a result we know the l.h.s. of eq.(79) thus providing the second iteration of the Volterra equation in terms of hypergeometric functions and derivatives thereof. Explicitly we find in eq.(54)

$$\begin{aligned} \text{tr}(S_2^+ - S_2^-)M^0 = & \\ 4\frac{\sin \pi \lambda_1}{\lambda_1} \sin \pi \lambda_\infty B_1^{(1)} B_2^{(1)} B_1^{(2)} B_2^{(2)} \{ (q_2 - \tilde{q}_2) \cdot A^{-1} \cdot (\hat{T}(1, 1) - \hat{T}(2, 2)) - \lambda_\infty^2 \lambda_1 c^2 \} . & \end{aligned} \quad (81)$$

The computation of the term  $-\text{tr}(S_1^+ - S_1^-)M^0 S_1^-$  in eq.(54) requires simply the knowledge of  $S_1^\pm$  which we have already computed and it cancels the term  $-\lambda_\infty^2 \lambda_1 c^2$  in the curly brackets in the above equation. To summarize the equation for  $C''(0)$  is given by

$$0 = (q_2 - \tilde{q}_2) \cdot A^{-1} \cdot (\hat{T}(1, 1) - \hat{T}(2, 2)) . \quad (82)$$

Due to the structure of  $\hat{T}(j, k)$ , see eq.(66), the vector  $\hat{T}(1, 1) - \hat{T}(2, 2)$  has a single entry different from zero and we have for  $C''(0) \equiv [xC_L(x)]''_{x=0} - 2C'(0) - 2C(0)$

$$\begin{aligned} C''(0) = & -\frac{(\delta_\infty + \delta_\nu - \delta_1)[C''(0) - 3\delta + c^2(2\delta_\nu + \delta_\infty - \delta_1)]}{\delta_\nu} \\ & - \frac{(C(0) + 3\delta - 3c^2\delta_\nu)[3\delta_1^2 + 3\delta_\nu^2 + 3\delta_\infty(1 + \delta_\infty) + \delta_\nu(3 + 2\delta_\infty) - 3\delta_1(1 + 2\delta_\nu + 2\delta_\infty)]}{\delta_\nu(3 + 4\delta_\nu)} \end{aligned} \quad (83)$$

where  $C(0)$ ,  $C'(0)$  and  $c$  are given respectively by eqs.(3,52,74). The value of  $C''(0)$  agrees with the one obtained in [6] and [1] by taking the  $b \rightarrow 0$  limit of the conformal blocks thus providing strong support to the exponentiation hypothesis.

The procedure can also be pushed to higher order even if we need a systematic organization of the mixed contributions.

Integrals similar to those discussed here appear in the accessory parameter problem for the torus, dealt with in [14]. There the logarithmic part was derived to all orders and compared with success with the saddle point prediction on the quantum theory, while the presence of integrals of the above mentioned type hampered the analytical evaluation of the  $q$  term of the expansion,  $q$  being the nome of the torus. Now we have the possibility of computing analytically not only the  $q$  term but also the  $q^2$  term.

## 5 Conclusions

In this paper we developed, for the monodromy problem considered in papers [1, 2] an analytical technique to compute the expansion of the accessory parameter in term of the invariant cross ratio directly, without taking the limit of the quantum conformal blocks for infinite central charge. In the first order computation it is shown how the integrals

containing products of two hypergeometric functions, which appear in the first iteration of the Volterra equation can be computed analytically in terms of  $\Gamma$ -functions and derivatives thereof, i.e.  $\psi$ -functions. The method is to exploit the transformation properties of the kernel under  $SL(2C)$  and the  $-1/2$ -form nature of the solutions.

The computation to second order involves double-integrals of products of four hypergeometric functions, and we show how also these can be computed analytically. In such second order computation we need a transformation which at the infinitesimal level is akin to the  $L_{-2}$  generator of the Virasoro algebra. The transformation is not one-to-one on the complex plane but is well defined in the region needed for our computations.

Both our first and second order results agree with the ones obtained from the classical limit of the quantum conformal blocks under the exponentiation hypothesis [6, 1] and thus they lend strong support to such exponentiation hypothesis in the classical  $b \rightarrow 0$  limit.

With regard to the extension to higher orders we have shown how the fundamental matrix elements  $N_m(j, k)$  can all be computed by a simple iterative procedure. As it happens already to second order in the  $n$ -th order computation one has both direct contribution from the  $n$ -order term in the expansion of the kernel and mixed contributions due to lower order expansion. One should need a systematic organization of such mixed contribution to go to arbitrary order.

In the full treatment of the torus with one source [14], integrals of product of hypergeometric function, similar to those which we discussed above, appeared. The possibility of computing them analytically will allow an extension of the results reported in [14].

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